Locally Anisotropic Wormholes and Flux Tubes in 5D Gravity

Sergiu I. Vacaru [*]

Institute for Space Sciences, B. O. Box: MG-23, RO76900, Măgurele, Bucharest, Romania

D. Singleton [†]

Dept. of Phys., CSU Fresno, 2345 East San Ramon Ave. M/S 37 Fresno, CA 93740-8031, USA

Vitalie A. Boţan [‡] and Denis A. Doţenco [§]

Faculty of Physics, State University of Moldova 60 Mateevici str., Chişinău MD2009, Republic of Moldova

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Abstract

In this article we examine a class of wormhole and flux tube like solutions to 5D vacuum Einstein equations. These solutions possess generic local anisotropy, and their local isotropic limit is shown to be conformally equivalent to the spherically symmetric 5D solutions of Ref. [1]. The anisotropic solutions investigated here have two physically distinct signatures: First, they can give rise to angular–dependent, anisotropic "electromagnetic" interactions. Second, they can result in a gravitational running of the "electric" and "magnetic" charges of the solutions. This gravitational running of the electromagnetic charges is linear rather than logarithmic, and could thus serve as an indirect signal for the presence of higher dimensions. The local anisotropy of these solutions is modeled using the technique of anholonomic frames with respect to which the metrics are diagonalized. If holonomic coordinate frames were used then such metrics would have off–diagonal components.

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[*]E-Mails : sergiu_vacaru@yahoo.com, sergiuvacaru@venus.nipne.ro

[†]E-Mail : dougs@csufresno.edu

[‡]E-Mail : botsan_v@yahoo.com

[§]E-Mail : dotsencodenis@yahoo.com, dotsenco@mail.com

I. INTRODUCTION

Recently one of the authors [2] studied a class of solutions to Einstein's equations in three (3D) and four (4D) dimensions which had generic local anisotropy (*e.g.* static black hole and cosmological solutions with ellipsoidal or toroidal symmetry). These solutions were obtained and studied using the method of moving anholonomic frames with an associated nonlinear connection structure on (pseudo) Riemannian spaces. This technique was formally developed in Refs. [3] where it was used to investigate locally anisotropic (super) string theory and supergravity models with anisotropic structures. In this paper we apply this method of anholonomic frames to Kaluza–Klein theory in order to construct metrics which describe locally anisotropic wormhole and flux tube solutions for the 5D vacuum Einstein equations.

In 4D and higher dimensional gravity there are well known spherical symmetric solutions which describe black holes and wormholes [4]. These solutions have diagonal metrics. Considering metrics in higher dimensional gravity which have non–zero off–diagonal components can lead to interesting physical consequences. Salam, Strathee and Perracci [5] showed that including off–diagonal components in higher dimensional metrics is equivalent to including gauge fields, and concluded that geometrical gauge fields can act as possible sources of exotic matter necessary for the construction of a wormhole.

In Refs. [6,7] various locally isotropic solutions with off–diagonal metric components were obtained for 5D vacuum Einstein equations. These solutions were similar to spherically symmetric 4D wormhole and/or flux tube metrics with "electric" and/or "magnetic" fields running along the throat of the wormhole. These "electromagnetic" fields arose as a consequence of the off–diagonal elements of the metric. By varying certain free parameters of the off–diagonal elements of the 5D metric, it was possible to change the relative strengths of the fields in the wormholes throat, and to change the longitudinal and transverse size of the wormhole's throat. Also for certain values of these parameters the 5D solutions could be related to 4D Reissner–Nordstrom solutions.

In the present work we use the method of anholonomic frames with associated nonlinear connections to construct locally *anisotropic* wormhole and flux tube solutions that reduce to the solutions of Ref. [1] in the isotropic limit. These solutions have off–diagonal (with respect to coordinate frames) metrics with generic, anholonomic vacuum polarizations of the 5D vacuum gravitational fields. These solutions exhibit two, distinct features depending on how the anisotropies are introduced. For anisotropies in the extra spatial dimension, χ, the solutions have a gravitational running or scaling of the "electromagnetic" charges. This gravitational scaling is linear rather than logarithmic, and might therefore provide an indirect signature for the presence of extra dimensions. For anisotropies in the axial angle, φ, we find that an angle–dependent, anisotropic "electromagnetic" interaction results for a test particle which moves in the background field of the solution.

The paper is organized as follows : In Sec. 2 we review the results for locally isotropic wormhole and flux tube solutions. In Sec. 3 we present the necessary geometric background on anholonomic frames and associated nonlinear connections in 5D (pseudo) Riemannian geometry. In Sec. 4 we construct and analyze a new class of 5D metrics describing anisotropic wormhole and flux tube solutions of the vacuum Einstein equations. In Sec. 5 we give an example of a wormhole like solution where the extra spatial dimension is handled by an exponential warp factor [8] rather than being compactified as in standard Kaluza-Klein

theory. In Sec. 6 we summarize and discuss the main conclusions of the paper.

II. 5D LOCALLY ISOTROPIC WORMHOLE LIKE SOLUTIONS

Here we give a brief review of the wormhole and flux tube solutions (DS-solutions) of Refs. [1,7]. The ansatz for the spherically symmetric 5D metric is taken as

$$ds^2_{(DS)} = e^{2\nu(r)}dt^2 - dr^2 - a(r)(d\theta^2 + \sin^2\theta d\varphi^2)$$
$$- r_0^2 e^{2\psi(r) - 2\nu(r)} \left[dx^5 + \omega(r)dt + n\cos\theta d\varphi \right]^2, \tag{1}$$

x^5 is the 5th extra, spatial coordinate; r, θ, φ are 3D spherical coordinates; n is an integer; $r \in \{-R_0, +R_0\}$ ($R_0 \le \infty$) and r_0 is a constant. All functions $\nu(r), \psi(r)$ and $a(r)$ are taken to be even functions of r satisfying $\nu'(0) = \psi'(0) = a'(0) = 0$. The coefficient $\omega(r)$ is treated as the t–component of the electromagnetic potential and $n\cos\theta$ is the φ-component. These electromagnetic potentials lead to this metric having radial Kaluza-Klein "electrical" and "magnetic" fields. The metric (1) is assumed to solve the 5D Einstein vacuum equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0, \tag{2}$$

Greek indices $\alpha, \beta, ...$ run over $0, 1, 2, 3, 4$. The 5D Kaluza-Klein "electric" field is

$$E_{KK} = r_0\omega' e^{3\psi - 4\nu} = \frac{q_0}{a(r)} \tag{3}$$

with the "electric" charge

$$q_0 = r_0\omega'(0) \tag{4}$$

which can be parametrized as

$$q_0 = 2\sqrt{a(0)}\sin\alpha_0, \tag{5}$$

The corresponding dual, "magnetic" field $H_{KK} = Q_0/a(r)$ with "magnetic" charge $Q_0 = nr_0$ is parametrized as

$$Q_0 = 2\sqrt{a(0)}\cos\alpha_0,$$

The following relation

$$\frac{(q_0^2 + Q_0^2)}{4a(0)} = 1 \tag{6}$$

relates the "electric" and "magnetic" charges. Later when we consider anisotropic versions of the metric in Eq. (1) the charges will have a dependence on φ or χ. Eq. (6) will still hold for these coordinate dependent charges, and thus enforces a definite relationship between the "electric" and "magnetic" charges. The solution in [1] satisfied the boundary conditions $a(0) = 1, \psi(0) = \nu(0) = 0$. As the free parameters of the metric are varied there are five classes of solutions with the properties:

1. $Q = 0$ or $H_{KK} = 0$, a wormhole–like "electric" object;

2. $q = 0$ or $E_{KK} = 0$, a finite "magnetic" flux tube;

3. $H_{KK} = E_{KK}$, an infinite "electromagnetic" flux tube;

4. $H_{KK} < E_{KK}$, a wormhole–like "electromagnetic" object;

5. $H_{KK} > E_{KK}$, a finite, "magnetic–electric" flux tube.

Here we will generalize these solutions (1) to locally anisotropic configurations using anholonomic frames.

In order to simplify the procedure of construction of new classes of solutions with generic local anisotropy it convenient to introduce a new 5th coordinate χ by a transform $x^5 \to \chi$ so that the coordinates are related via

$$dx^5 + n\cos\theta d\varphi = d\chi + n\cos\theta d\theta$$

this can be accomplished by taking $\chi = x^5 - \int[\mu(\theta, \varphi)]^{-1}d\xi(\theta, \varphi)$ with

$$\frac{\partial\xi}{\partial\varphi} = -\mu n\cos\theta, \qquad \frac{\partial\xi}{\partial\theta} = \mu n\cos\theta$$

and

$$\mu(\theta, \varphi) = \exp(\theta - \varphi)|\cos\theta|^{-1}.$$

so that the mixed partials of ξ are consistent. With respect to the new extra dimensional co-ordinate ξ the A_φ component of the electromagnetic potential is exchanged for a component A_θ and the metric interval (1) is rewritten as

$$
\begin{aligned}
ds^2_{(DS,new)} = {} & e^{2\nu(r)}dt^2 - dr^2 - a(r)(d\theta^2 + \sin^2\theta d\varphi^2) \\
& - r_0^2 e^{2\psi(r)-2\nu(r)}\left[d\chi + \omega(r)dt + n\cos\theta d\theta\right]^2,
\end{aligned}
\tag{7}
$$

This allows us to treat the (t, r, θ) coordinates as holonomic and the (φ, χ) coordinates as anholonomic.

III. ANHOLONOMIC FRAMES AND LOCAL ANISOTROPY

Let us consider a 5D pseudo–Riemannian spacetime of signature $(+, -, -, -, -)$ and denote the local coordinates $u^\alpha = (x^i, y^a)$, – or more compactly $u = (x, y)$ – where the Greek indices are conventionally split into two subsets x^i and y^a labeled respectively by Latin indices of type $i, j, k, \ldots = 1, 2, \ldots n$ and $a, b, \ldots = 1, 2, \ldots, m$, with $n + m = 5$. The local coordinate bases, $\partial_\alpha = (\partial_i, \partial_a)$, and their duals, $d^\alpha = (d^i, d^a)$, are defined respectively as

$$\partial_\alpha \equiv \frac{\partial}{du^\alpha} = (\partial_i = \frac{\partial}{dx^i}, \partial_a = \frac{\partial}{dy^a}) \tag{8}$$

and

$$d^\alpha \equiv du^\alpha = (d^i = dx^i, d^a = dy^a). \tag{9}$$

In Kaluza–Klein theories (see Ref. [9] for a review) one often parameterizes the metric as

$$g_{\alpha\beta} = \begin{bmatrix} g_{ij}(u) + N_i^a(u)N_j^b(u)h_{ab}(u) & N_j^e(u)h_{ae}(u) \\ N_i^e(u)h_{be}(u) & h_{ab}(u) \end{bmatrix} \tag{10}$$

with the coefficients, $N_i^a(u)$ *etc.*, given with respect to a local coordinate basis. A 5D metric $g^{(5)}$ with coefficients (10) splits into a block $(n \times n) + (m \times m)$ form

$$\delta s^2 = g_{ij}(x, y)\, dx^i dx^j + h_{ab}(x, y)\, \delta y^a \delta y^b \tag{11}$$

Instead of using the coordinate bases (8) and (9) one can introduce locally anisotropic frames (bases)

$$\delta_\alpha \equiv \frac{\delta}{du^\alpha} = (\delta_i = \partial_i - N_i^b(u)\, \partial_b, \partial_a = \frac{\partial}{dy^a}) \tag{12}$$

and

$$\delta^\alpha \equiv \delta u^\alpha = (\delta^i = dx^i, \delta^a = dy^a + N_k^a(u)\, dx^k). \tag{13}$$

The main 'trick' of the anholonomic frames method for integrating the Einstein equations [2,3] is to find N_j^a's such that the block matrices g_{ij} and h_{ab} are diagonalized. This greatly simplifies computations. With respect to these anholonomic frames the partial derivatives are N–elongated (locally anisotropic).

This topic of anholonomic frames is related to the geometry of moving frames and non-linear connection structures on manifolds and vector bundles. The idea is to take the coefficients $N_i^a(x, y)$ as defining a nonlinear connection (an N–connection) structure with the curvature (N–curvature)

$$\Omega_{ij}^a \doteq \delta_j N_i^a - \delta_i N_j^a$$

This induces a global decomposition of the 5D pseudo–Riemannian spacetime into holonomic (horizontal, h) and anholonomic (vertical, v) directions. In a preliminary form the concept of N–connections was applied by E. Cartan in his approach to Finsler geometry [10] and a rigorous definition was given by Barthel [11] (Ref. [12] gives a modern approach to the geometry of N–connections, and to generalized Lagrange and Finsler geometry). As a particular case one obtains the linear connections if $N_i^a(x, y) = \Gamma_{bi}^a(x)\, y^a$.

A more surprising result is that N–connection structures can be naturally defined on (pseudo) Riemannian spacetimes [2] by associating them with some anholonomic frame fields (vielbeins) of type (12) satisfying the relations

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w^\gamma{}_{\alpha\beta} \delta_\gamma$$

where the anholonomy coefficients $w^\gamma{}_{\alpha\beta}$ are computed as follows

$$w^k{}_{ij} = 0\,,\ w^k{}_{aj} = 0\,,\ w^k{}_{ia} = 0\,,\ w^k{}_{ab} = 0\,,\ w^c{}_{ab} = 0,$$
$$w^a{}_{ij} = -\Omega_{ij}^a\,,\ w^a{}_{bj} = -\partial_b N_j^a\,,\ w^b{}_{ia} = \partial_a N_i^b. \tag{14}$$

One says that the N–connection coefficients model a locally anisotropic structure on space-time (an locally anisotropic spacetime) when the partial derivative operators and coordinate differentials, (8) and (9), are respectively changed into N–elongated operators (12) and (13).

A linear connection $D_{\delta_\gamma}\delta_\beta = \Gamma^\alpha_{\beta\gamma}(x,y)\delta_\alpha$ is compatible with the metric $g_{\alpha\beta}$ and N–connection structure on a 5D pseudo–Riemannian spacetimes, if $D_\alpha g_{\beta\gamma} = 0$. The linear connection is parametrized by irreducible h–v–components,

$$\Gamma^\alpha_{\beta\gamma} = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\right),$$

where

$$L^i{}_{jk} = \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right),$$
$$L^a{}_{bk} = \partial_b N^a_k + \frac{1}{2}h^{ac}\left(\delta_k h_{bc} - h_{dc}\partial_b N^d_k - h_{db}\partial_c N^d_k\right), \tag{15}$$
$$C^i{}_{jc} = \frac{1}{2}g^{ik}\partial_c g_{jk}, \ \ C^a{}_{bc} = \frac{1}{2}h^{ad}\left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}\right)$$

This defines a canonical linear connection (as distinguished from an N–connection) which is similar to the metric connection introduced by Christoffel symbols in the case of holonomic bases.

The anholonomic coefficients $w^\gamma{}_{\alpha\beta}$ and N–elongated derivatives give nontrivial coefficients for the torsion tensor, $T(\delta_\gamma, \delta_\beta) = T^\alpha_{\beta\gamma}\delta_\alpha$, where

$$T^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta} + w^\alpha_{\beta\gamma}, \tag{16}$$

and for the curvature tensor, $R(\delta_\tau, \delta_\gamma)\delta_\beta = R_\beta{}^\alpha{}_{\gamma\tau}\delta_\alpha$, where

$$R_\beta{}^\alpha{}_{\gamma\tau} = \delta_\tau\Gamma^\alpha_{\beta\gamma} - \delta_\gamma\Gamma^\alpha_{\beta\tau} + \Gamma^\varphi_{\beta\gamma}\Gamma^\alpha_{\varphi\tau} - \Gamma^\varphi_{\beta\tau}\Gamma^\alpha_{\varphi\gamma} + \Gamma^\alpha_{\beta\varphi}w^\varphi_{\gamma\tau}. \tag{17}$$

We emphasize that the torsion tensor on (pseudo) Riemannian spacetimes is induced by anholonomic frames, whereas its components vanish with respect to holonomic frames. All tensors are distinguished (d) by the N–connection structure into irreducible h–v–components, and are called d–tensors. For instance, the torsion, d–tensor has the following irreducible, nonvanishing, h–v–components,

$$T^\alpha_{\beta\gamma} = \{T^i{}_{jk}, C^i{}_{ja}, S^a{}_{bc}, T^a{}_{ij}, T^a{}_{bi}\},$$

where

$$T^i_{.jk} = T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = C^i_{.ja}, \quad T^i_{aj} = -C^i_{ja}, T^i_{.ja} = 0,$$
$$T^a_{.bc} = S^a_{.bc} = C^a_{bc} - C^a_{cb}, T^a_{.ij} = -\Omega^a_{ij}, \quad T^a_{.bi} = \partial_b N^a_i - L^a_{.bi}, \quad T^a_{.ib} = -T^a_{.bi}$$

(the d–torsion is computed by substituting the h–v–components of the canonical d–connection (15) and anholonomic coefficients (14) into the formula for the torsion coefficients (16)). The curvature d-tensor has the following irreducible, non-vanishing, h–v–components

$$R_\beta{}^\alpha{}_{\gamma\tau} = \{R^i_{h.jk}, R^a_{b.jk}, P^i_{j.ka}, P^c_{b.ka}, S^i_{j.bc}, S^a_{b.cd}\},$$

where

$$R^i_{h.jk} = \delta_k L^i_{.hj} - \delta_j L^i_{.hk} + L^m_{.hj} L^i_{mk} - L^m_{.hk} L^i_{mj} - C^i_{.ha}\Omega^a_{.jk},$$
$$R^a_{b.jk} = \delta_k L^a_{.bj} - \delta_j L^a_{.bk} + L^c_{.bj} L^a_{.ck} - L^c_{.bk} L^a_{.cj} - C^a_{.bc}\Omega^c_{.jk},$$
$$P^i_{j.ka} = \partial_a L^i_{.jk} + C^i_{.jb} T^b_{.ka} - (\delta_k C^i_{.ja} + L^i_{.lk} C^l_{.ja} - L^l_{.jk} C^i_{.la} - L^c_{.ak} C^i_{.jc}),$$
$$P^c_{b.ka} = \partial_a L^c_{.bk} + C^c_{.bd} T^d_{.ka} - (\delta_k C^c_{.ba} + L^c_{.dk} C^d_{.ba} - L^d_{.bk} C^c_{.da} - L^d_{.ak} C^c_{.bd}),$$
$$S^i_{j.bc} = \partial_c C^i_{.jb} - \partial_b C^i_{.jc} + C^h_{.jb} C^i_{.hc} - C^h_{.jc} C^i_{hb},$$
$$S^{.a}_{b.cd} = \partial_d C^a_{.bc} - \partial_c C^a_{.bd} + C^e_{.bc} C^a_{.ed} - C^e_{.bd} C^a_{.ec}$$

(the d–curvature components are computed in a similar fashion by using the formula for curvature coefficients (17)). The Ricci tensor $R_{\beta\gamma} = R^{\alpha}_{\beta\ \gamma\alpha}$ has the d–components

$$R_{ij} = R^{.k}_{i.jk}, \quad R_{ia} = -^2 P_{ia} = -P^{.k}_{i.ka}, \qquad R_{ai} =^1 P_{ai} = P^{.b}_{a.ib}, \quad R_{ab} = S^{.c}_{a.bc}. \tag{18}$$

In general, since $^1P_{ai} \neq \ ^2P_{ia}$, the Ricci d-tensor is non-symmetric (this could be with respect to anholonomic frames of reference). The scalar curvature of the metric d–connection, $\overleftarrow{R} = g^{\beta\gamma} R_{\beta\gamma}$, is computed as

$$\overleftarrow{R} = G^{\alpha\beta} R_{\alpha\beta} = \widehat{R} + S, \tag{19}$$

where $\widehat{R} = g^{ij} R_{ij}$ and $S = h^{ab} S_{ab}$.

By substituting (18) and (19) into the 5D Einstein equations (2) we obtain a system of vacuum, gravitational field equations with mixed holonomic–anholonomic degrees of freedom with an N–connection structure,

$$R_{ij} - \frac{1}{2}\left(\widehat{R} + S\right) g_{ij} = 0, \tag{20}$$
$$S_{ab} - \frac{1}{2}\left(\widehat{R} + S\right) h_{ab} = 0,$$
$$^1P_{ai} = 0, \ ^2P_{ia} = 0.$$

The definition of matter sources with respect to locally anisotropic frames is dealt with in Refs. [2]. In this paper we deal only with vacuum 5D, locally, anisotropic gravitational equations.

IV. 5D ANISOTROPIC WORMHOLE LIKE SOLUTIONS

The 5D metric (7) lends itself very naturally to the funfbein (5D frame) formalism. We are interested in constructing and investigating generalizations of this metric to anisotropic configurations with two anholonomic coordinates : φ and χ. An ansatz of type (11), written in terms of an locally anisotropic bases of type (13), is taken as

$$\delta s^2 = dt^2 - b\left(\zeta\right) d\zeta^2 - c(\zeta)d\theta^2 - h_4\left(\zeta, \theta, y^1\right) (\delta y^1)^2 - h_5\left(\zeta, \theta, y^1\right) (\delta y^2)^2 \tag{21}$$

for $x^1 = t, x^2 = \zeta, x^3 = \theta, y^1 = v = \varphi(\text{ or } v = \chi), y^2 = p = \chi(\text{ or } p = \varphi)$. For nontrivial values of $\nu(r)$ the coordinate r is related to ζ via $\zeta = \int exp[-\nu(r)]dr$. If $\nu(r)$ is fixed (e.g. $\nu(r) = 0$) then we can set $\zeta = r$. The N–elongated differentials are

$$\delta y^1 = dv + w_2\left(\zeta, \theta, v\right) d\zeta + w_3\left(\zeta, \theta, v\right) d\theta, \tag{22}$$
$$\delta y^2 = dp + n_2\left(\zeta, \theta, v\right) d\zeta + n_3\left(\zeta, \theta, v\right) d\theta,$$

i.e. the coefficients of the N–connection are parametrized as $N_1^1 = 0, N_{2,3}^1 = w_{2,3}\left(\zeta, \theta, v\right), N_1^2 = 0, N_{2,3}^2 = n_{2,3}\left(\zeta, \theta, v\right)$. We will consider anisotropic dependencies on the coordinate $y^1 = v$ (a similar construction holds if we consider dependencies on $y^2 = p$).

The nontrivial components of the 5D vacuum Einstein equations (20) for the metric (21) are computed as

$$c'' - \frac{1}{2c}c'^2 - \frac{1}{2b}c'b' = 0, \tag{23}$$

$$\ddot{h}_5 - \frac{1}{2h_5}\dot{h}_5^2 - \frac{1}{2h_4}\dot{h}_5\dot{h}_4 = 0, \tag{24}$$

$$\beta w_{2,3} + \alpha_{2,3} = 0, \tag{25}$$

$$\ddot{n}_{2,3} + \gamma \dot{n}_{2,3} = 0, \tag{26}$$

where

$$\alpha_2 = \frac{\dot{h}_5}{2}\left(\frac{h_4'}{h_4} + \frac{h_5'}{h_5}\right) - \dot{h}_5', \qquad \alpha_3 = \frac{\dot{h}_5}{2}\left(\frac{h_4^*}{h_5} + \frac{h_5^*}{h_5}\right) - \dot{h}_5^*,$$

$$\beta = \ddot{h}_5 - \frac{\dot{h}_5^2}{2h_5} - \frac{\dot{h}_4}{\dot{h}_5}2h_4, \qquad \gamma = \frac{3}{2}\frac{\dot{h}_5}{h_5} - \frac{\dot{h}_4}{h_4}.$$

For simplicity, the partial derivatives are denoted as $\dot{h}_4 = \partial h_4/\partial v, h_5' = \partial h_5/\partial \zeta$ and $h_5^* = \partial h_5/\partial \theta$.

A. Anisotropic solutions of 5D vacuum Einstein equations

It is possible to find the general solutions of the Eqs. (23), (24) and (26). Eq. (23) relates two functions $c(\zeta)$ and $b(\zeta)$. For a prescribed value of $b(\zeta)$ the general solution is

$$c(\zeta) = \left[c_0 + c_1 \int \sqrt{|b(\zeta)|}d\zeta\right]^2 \tag{27}$$

where c_0 and c_1 are integration constants, which could depend parametrically on the θ variable. Alternatively for a prescribed $c = c(\zeta)$ the general solution is

$$b(\zeta) = b_0[c'(\zeta)]^2/c(\zeta) \tag{28}$$

with $b_0 = const$. In a similar fashion we can construct the general solution of (24),

$$h_5\left(\zeta, \theta, v\right) = \left[h_{5[1]}(\zeta, \theta) + h_{5[0]}(\zeta, \theta)\int \sqrt{|h_4(\zeta, \theta, v)|}dv\right]^2, \tag{29}$$

where $h_{5[0]}(\zeta, \theta)$ and $h_{5[1]}(\zeta, \theta)$ are functions of holonomic variables, or

$$h_4\left(\zeta, \theta, v\right) = h_{4[0]}\left(\zeta, \theta\right)[\dot{h}_5(\zeta, \theta, v)]^2/h_5(\zeta, \theta, v), \tag{30}$$

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where $h_{5[0]}(\zeta, \theta)$ is defined from some compatibility conditions with the local isotropic limit.

Having defined the functions $h_5(\zeta, \theta, v)$, and $h_4(\zeta, \theta, v)$ we can compute $\alpha_{2,3}$ and β, and express the solutions of equations (25) as

$$w_{2,3} = -\alpha_{2,3}/\beta. \tag{31}$$

In a similar fashion, after two integrations of the anisotropic coordinate v we can find the general solution to equations (26)

$$n_{2,3}(\zeta, \theta, v) = n_{2,3[0]}(\zeta, \theta) + n_{2,3[1]}(\zeta, \theta) \int \left(h_4(\zeta, \theta, v)/h_5^{3/2}(\zeta, \theta, v) \right) dv, \tag{32}$$

where $n_{2,3[0]}(\zeta, \theta)$ and $n_{2,3[1]}(\zeta, \theta)$ are defined from boundary conditions.

Equations (25) are second order linear differential equations of the anisotropic coordinate v. The functions $h_{4,5}$ can be thought of as parameters; the explicit form of the solutions of (25) depends on the values of these parameters. Compatibility with the locally isotropic limit (*i.e.* when $w_{2,3} \to 0$,) is possible if the conditions $\alpha_{2,3} = 0$ are satisfied for isotropic configurations. Methods for constructing solutions to such equations, when boundary conditions at some $v = v_0$ are specified, can be found in [13]. We will assume that we can always define the functions $w_{2,3}(\zeta, \theta, v)$ for given $h_5(\zeta, \theta, v)$ and $h_4(\zeta, \theta, v)$; in this case equations (25) can be treated as linear algebraic equations for $w_{2,3}$ with the coefficients $\alpha_{2,3}$ and β determined using the solutions of (24).

B. Locally anisotropic generalizations of DS–metrics

We now consider two particular types of solutions, which in the locally isotropic limit can be connected with the DS–metric (7). These anisotropic solutions of the 5D vacuum Einstein equations will satisfy

$$b(\zeta) = 1, \quad c(\zeta) = a(\zeta), \quad h_4(\zeta, \theta) = a(\zeta) \sin^2 \theta, \tag{33}$$

as do the locally isotropic solutions of (1), but now r_0 is not a constant, but is a function like $\widehat{r}_0^2(v)$. The function h_5 is parametrized as

$$h_5 = \exp[2\psi(\zeta)]\widehat{r}_0^2(v) \tag{34}$$

where

$$\widehat{r}_0^2(v) = r_{0(0)}^2(1 + \varepsilon v)^2 \tag{35}$$

The "renormalization" constant, ε, is obtained either from experiment or from some quantum gravity model. In the locally isotropic limit, $(\varepsilon v \to 0)$ we recover the DS–metric.

In order to analyze the locally isotropic limit $(\varepsilon v \to 0)$ where the frame functions transforms like

$$a(\zeta) \to a(r), \quad \nu(\zeta) \to \nu(r), \quad \psi(\zeta) \to \psi(r)$$

we impose the limiting condition $[\dot{h}_5(\zeta, \theta, v)]^2/h_5(\zeta, \theta, v) \to \exp[2\psi(r)]$ in (30) which is satisfied for $\varepsilon v \to 0$ if $\varepsilon = 1/2r_{0(0)}^2$ and choose the function $h_{4[0]}(\zeta, \theta)$ from (30) so that

$$h_{4[0]}(\zeta, \theta) \exp[2\psi(\zeta)] \to a(r) \sin^2 \theta.$$

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It should be emphasized that (27) gives a dependence like

$$c(r) = a(r) \exp[\nu(r)] = (c_0 + c_1 r)^2$$

which holds for a fixed conformal factor $\exp[\nu(r)]$. Thus in the locally isotropic limit metric (21) becomes the DS–metric (7) multiplied by a conformal factor, $\exp[-2\nu(r)]ds^2_{(DS)}$.

The 5D gravitational vacuum polarization renormalizes the charge defined in Eq. (4) as

$$q(v) = \hat{r}_0(v)\omega'(0) = r_{0(0)}(1 + \varepsilon v)\omega'(0). \tag{36}$$

The angular parametrization (5) also becomes locally anisotropic,

$$q(v) = 2\sqrt{a(0)} \sin \alpha(v), \tag{37}$$

and the formula for the "electric" field (3) transforms into

$$E_{KK} = \frac{q(v)}{a(r)}. \tag{38}$$

For the case when $v = \chi$ Eq. (36) implies that the "electric" charge of this solution exhibits a linear scaling with respect to the extra spatial coordinate. This can be contrasted with the standard quantum field theory calculation where the electric coupling grows logarithmically as the distance scale decreases. Also in the present case the scaling of the "electric" charge arises classically as a result of the anisotropy from the extra spatial coordinate. In contrast the usual logarithmic scaling of electric charge is a *quantum* field theory result. In the standard Kaluza-Klein scenario the extra spatial coordinate is taken to be compactified on the order of the Planck length (*i.e.* $v = \chi \simeq L_{Planck}$). Unless ε is unnaturally large this implies $\varepsilon\chi << 1$ making this linear running of the "electric" charge unobservable. However, if the extra dimension(s) have a larger size [14] then such a linear, gravitational running of the charges, with respect to the size of the extra dimension(s), could be observable, and could provide an indirect indication of the presence of higher dimensions. The renormalization of the magnetic charge, $Q_0 \rightarrow Q(v)$, can be obtained using the renormalized "electric" charge in relationship (6) and solving for $Q(v)$. The form of (6) implies that the running of the "magnetic" charge $Q(v)$ will be the opposite that of the "electric" charge, $q(v)$. For example, if $q(v)$ increases with v then $Q(v)$ will decrease. For small values of the "renormalization" constant, ε, the function $a(r)$ will take a form similar to that of the DS–solution.

In the case were the anisotropy arises from the φ coordinate one can see that the "electric" charge in Eq. (36) has a dependence on φ, which leads to the Kaluza-Klein electric field having an anisotropic φ dependence. A test charge placed in the background of such a solution would experience an anisotropic, φ–dependent interaction. Thus the anisotropies of this solution result in anisotropic interactions for the matter fields in 4D.

The usefulness of the anholonomic frames method in dealing with anisotropic configurations can be seen by writing out the ansatz metric (11), for the two classes of solutions (*i.e.* $v = \varphi$ or $v = \chi$) discussed above, in a coordinate frame (8).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 - w_2{}^2 h_4 - n_2{}^2 h_5 & -w_2 w_3 h_4 - n_2 n_3 h_5 & -w_2 h_4 & -n_2 h_5 \\ 0 & -w_3 w_2 h_4 - n_2 n_3 h_5 & -a - w_3{}^2 h_4 - n_3{}^2 h_5 & -w_3 h_4 & -n_3 h_5 \\ 0 & -w_2 h_4 & -w_3 h_4 & -h_4 & 0 \\ 0 & -n_2 h_5 & -n_3 h_5 & 0 & -h_5 \end{bmatrix}. \tag{39}$$

In this coordinate frame the ansatz metric $g_{\alpha\beta}$ has many non–diagonal terms which complicates the study of this metric in the coordinate frame. In contrast in the anholonomic frame the metric has only diagonal terms (see Eq.(39) which makes the system much easier to study. From the above coordinate frame form of the ansatz metric one can see that in the locally isotropic limit – $w_{2,3} \to 0$, $n_2 \to \omega(r)$ and $n_3 \to n\cos\theta$ – the DS–solution is recovered. For small anisotropies the solutions presented above are similar to the five classes of wormhole or flux tube solutions enumerated at the end of section II. Stronger anisotropic, vacuum, gravitational polarizations could result in nonlinear renormalization of the effective "electromagnetic" constants, and could change substantially the wormhole – flux tube configurations.

V. EXAMPLE : ANISOTROPIC WARPED WORMHOLE LIKE CONFIGURATION

Up to this point we have implicitly been working in the standard Kaluza-Klein scenario where the extra dimension, χ is assumed to be "curled up" or compactified. However, within the present set up it is possible to find wormhole like solutions which treat the 5th coordinate as in the Randall-Sundrum (RS) scheme [8] where an exponential warp factor multiples the four other coordinates (i.e. the RS metric has a form like $ds^2 = e^{-2k|\chi|}\eta_{\mu\nu}dx^\mu dx^\nu + d\chi^2$ which exponentially suppresses motion off of the 4D space). To this end we multiple the d–metric ansatz (21) by the conformal factor $\Omega_0^2(\zeta,\theta)\exp[-2k_\chi|\chi|]$ where k_χ is an arbitrary constant

$$\delta s^2 = \Omega_0^2(\zeta,\theta)\exp(-2k_\chi|\chi|)\left[dt^2 - b(\zeta)d\zeta^2 - c(\zeta)d\theta^2 - h_4(\zeta,\theta,\chi)(\check{\delta}\chi)^2 - h_5(\zeta,\theta,\chi)(\delta\varphi)^2\right] \tag{40}$$

where for the local coordinates are chosen $x^1 = t, x^2 = \zeta, x^3 = \theta, y^1 = v = \chi, y^2 = p = \varphi$, with $\check{\delta}\chi$ being an N–elongation of the χ–variable

$$\check{\delta}\chi = \delta\chi + \breve{w}_i(x^k,\chi)dx^i + \breve{w}_\varphi(x^k,\chi)\delta\varphi,$$

and $\delta\chi$ and $\delta\varphi$ are still N–elongated as in (22).

The 5D vacuum Einstein equations for this ansatz reduce to the system of equations (23)–(26) and some additional equations for the off–diagonal components of the Ricci d–tensor,

$$P_{4\check{i}} = -\psi_\chi\delta_{\check{i}}\ln\sqrt{|h_4|}, \tag{41}$$

where $\psi_\chi = \partial_\chi\ln|\Omega|$ for $\Omega = \Omega_0\exp(-2k_\chi|\chi|)$ and $\delta_{\check{i}} = \partial_{\check{i}} - \breve{w}_{\check{i}}$. The index $\check{i}$ runs over i and φ, i.e. $\breve{w}_{\check{i}} = (\breve{w}_i, \breve{w}_\varphi)$. A solution to these equations is

$$\Omega_0 = c^{1/4}(\zeta)|\sin\theta|, \qquad h_4 = c(\zeta)\sin^2\theta\exp(-4k_\chi|\chi|), \qquad h_5 = \exp[2\psi(\zeta)]\hat{r}_0^2(v) ;$$
$$\breve{w}_1 = 0, \qquad \breve{w}_2 = [\ln|\ln|c^{1/4}(\zeta)|]]', \qquad \breve{w}_3 = [\ln|\ln|\sin\theta|]]^*, \breve{w}_\varphi = 0. \tag{42}$$

The specific form for the functions $b(\zeta), c(\zeta)$, and h_5 is the same as in Eqs. (33), (34) and (35). This d–metric (40) with ansatz functions of the form (42) has the interesting property

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that it admits a warped conformal factor $\exp(-2k_\chi|\chi|)$ which is induced by a conformal transformation of the anisotropic wormhole like solution (21).

One of the chief consequences of the RS scheme is an *isotropic* deviation of Newtonian gravitational potential. In the original RS scenario this takes the form

$$V(r) = G_N \frac{m_1 m_2}{r} \left(1 + \frac{1}{r^2 k^2}\right)$$

for two interacting masses m_1, m_2. The deviation is isotropic since it only depends on r. Using a set up similar to our wormhole example of Eqs. (40)- (42), but applying it to the two brane scenario of Ref. [8] shows that anholonomic coordinates can give rise to "anisotropic" deviations of the Newtonian potential like [15]

$$V(r) = G_N \frac{m_1 m_2}{r} \left(1 + \frac{e^{-2k_y|y|}}{r^2 k^2}\right)$$

where y is a space-like coordinate and k_y is a constant to be measured or constrained experimentally. The details of such a construction can be found in Ref. [15]

VI. CONCLUSIONS

Solutions of wormhole–flux tube type are of fundamental importance in the construction of non–perturbative configurations in modern string theory, extra dimensional gravity and quantum–chromodynamics. Any solutions which aid in the physical understanding of such models are clearly beneficial. The solutions presented here have a number of features which achieve this aim. In addition this paper illustrates the usefulness of the method of moving anholonomic frames in finding new metrics and new types of locally anisotropic field interactions.

The class of solutions presented here demonstrates that wormholes and flux tubes are not only spherically symmetric, but can also have an anisotropic structure. Moreover these solutions demonstrate that the higher dimensional gravitational vacuum can be "polarized". This induces observable anisotropic effects in 4D spacetime: either an angle dependent interaction from the anisotropies in φ, or a running of the charges of the solution coming from the anisotropies in χ. Such solutions with generic local anisotropy are consistent, in the locally isotropic limit, with previously known wormhole metrics [1,6,7]. In section V we gave a wormhole like solution where the 5th coordinate was treated as in the RS scenario – motion off the 4D brane into the extra dimension is suppressed by an exponential warp factor. Our solution also exhibited an anisotropy through its dependence on the anholonomic variables. Applying these ideas to the Newtonian potential would lead to anisotropic deviations from the classical $1/r$ form [15]

A substantial new result of this paper is that the vacuum Einstein equations for 5D Kaluza–Klein theory can describe higher dimensional, locally anisotropic gravitational fields. These fields in turn induce anisotropic field interactions in 4D gravity and matter field theory. This emphasizes the importance of the problem of fixing ones reference system in gravity theories. This is not done by any dynamical field equations, but via physical considerations arising from the symmetries of the metric and/or the imposed boundary conditions. The

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analysis presented here shows that the anholonomy of higher dimensional frame (vielbein) fields transforms the 'lower' dimensional dynamics of interactions, basic equations, and their solutions, into locally anisotropic ones. This in part supports the results of 3D and 4D calculations [2] that locally anisotropic configurations and interactions can be handled in general relativity and its extra–dimensional variants on (pseudo) Riemannian spacetimes, through the application of the method of anholonomic frames.

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